Lecture 7 - Normal Forms

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1 Normal Forms

1.1 Negation Normal Form

A formula is in *negation normal form* if it does not contain the logical connectives $\rightarrow$ and if all negations are applied to atoms.

Any well-formed formula in the first-order logic can be transformed into a logically equivalent formula in negation normal form by exhaustively applying the transformation rules below (in any order):

1.	$(A \rightarrow B)$	$\rightarrowtail (\neg A \vee B)$
2a.	$\neg(\forall v)A$	$\rightarrowtail (\exists v)\neg A$
2b.	$\neg(\exists v)A$	$\rightarrowtail (\forall v)\neg A$
3.	$\neg\neg A$	$\rightarrowtail A$
4a.	$\neg(A \wedge B)$	$\rightarrowtail (\neg A \vee \neg B)$
4b	$\neg(A \vee B)$	$\rightarrowtail (\neg A \wedge \neg B)$

Figure 1: Negation Normal Form Transformation Rules

1.2 Renaming Variables

When a formula contains two or more variables that have the same name but are under the scope of different quantifiers we apply a renaming to the formula to give the variables under the scope of the innermost quantifier fresh variable names. For example we transform via renaming

$$(\forall x)P(x) \vee (\exists x)Q(x)$$

to

$$(\forall x)P(x) \vee (\exists y)Q(y)$$

.

1.3 Substitution

A *substitution* is a mapping from variables to terms. If Φ is a first-order formula and l is a variable and r is a terms then $\Phi_{l \mapsto r}$ denotes the formula Φ with all occurrences of l replaced by r.

Examples

$$isChildOf(x, Mary)_{x \mapsto Joe} \equiv isChildOf(Joe, Mary)$$

$$\exists x(P(x) \vee Q(x, a))_{x \mapsto b} \equiv \exists x(P(b) \vee Q(b, a))$$

1.4 Removing Existential Quantifiers

A formula with existential quantifiers can be transformed into an *equisatisfiable* formula without existential quantifiers via *Skolemization*.

To Skolemize a formula we remove the existential by applying the following rules:

Let Φ be a formula containing a variable v that is under the scope of a existential quantifier.

- If v is not under the scope of a universal quantifier then Φ is equisatisfiable to $\Phi_{v \mapsto a}$ where a is a fresh constant.
- If v is under the scope of universal quantifiers $(\forall x_1)...(\forall x_j)$ then Φ is equisatisfiable to $\Phi_{v \mapsto f(x_1, \ldots, x_j)}$ where f is a fresh function symbol.

Examples

$$\exists x(P(x) \vee Q(x, a)) \equiv \exists x(P(b) \vee Q(b, a)) \equiv P(b) \vee Q(b, a)$$

$$\forall y \exists x(P(y) \vee Q(x, a)) \equiv \forall y \exists x(P(y) \vee Q(f(y), a)) \equiv \forall y P(y) \vee Q(f(y), a)$$

1.5 Distributing $\vee$s Over $\wedge$s

We distribute $\vee$s over $\wedge$s using $A \vee (B \wedge C) \rightarrowtail (A \vee B) \wedge (A \vee C)$ and $(B \wedge C) \vee A \rightarrowtail (B \vee A) \wedge (C \vee A)$ for all formula A, B and C

Examples

$$P(x) \wedge (Q(y) \vee (R(z) \wedge S(x, z)))$$

$$\equiv P(x) \wedge (Q(y) \vee R(z)) \wedge (Q(y) \vee S(x, z))$$

1.6 Conjunctive Normal Form

A formula is in *conjunctive normal form* (CNF) if it is a conjunction of clauses such that negations are applied only to atoms and all variables are universally quantified.

A formula can be converted to an equisatisfiable formula in CNF by following the steps below.

1. Convert to negation normal form
2. Rename apart
3. Skolemize
4. Remove universal quantifiers
5. Distribute $\vee$s over $\wedge$s

Figure 2: CNF Conversion Steps